This is a closed book exam. You are supposed to know elementary trigonometric formulae and the exponential and logarithmic functions. I have also appended a list of useful formulae. These are adequate to solve all problems in this exam.

There are 5 questions. Full credit will correspond to the correct answers for the first 4 problems, for a total of 44 points. (This will be scaled appropriately to reflect the correct weight for this exam in your final grade.) Problem 5 will be treated as a bonus problem. If you work it out, it can help you make up points you may have lost elsewhere.

When you finish, please scan and send me a single pdf file of the exam, indicating on page 1 how many pages you are sending, so I will know if anything is missing in transmission. I expect the answers back by 2:10 PM at the latest. Answers returned after 2:10 PM will not be considered.

Problem 1 (*12 points*)

Consider a rail bent into a semicircular shape, of radius R, and placed on the ground as shown in figure. A disk of mass M and radius a can roll without slipping on the inside of this rail. (The motion is effectively one dimensional.)
a) Obtain the Lagrangian for the motion of the disk. (Moment of inertia of disk is $\frac{1}{2}Ma^2$.)
b) Obtain the Hamiltonian and the canonical equations of motion.
(*Caution*: The distance traced out by the center of mass of the disk is different from the distance it rolls because a is not negligible.)
c) Determine the frequency for small oscillations around the equilibrium point if $a = R/3$.
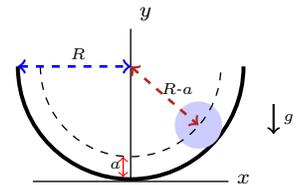
Solution
a) Taking the line from the center of the circular rail to the center of the disk to be at an angle φ from the vertical line, we find the coordinates of the center of mass as

$$(X, Y) = (R - a)\sin\varphi, \ -(R - a)\cos\varphi)$$

Let the angle turned by the wheel as it rolls be designated as α. The distance covered in the rolling motion, traced out on the inside of the rail, is $s = a\alpha$. This should equal the distance along the rail, i.e., $R\varphi$ if there is no slipping. Thus we have $a\alpha = R\varphi$. The kinetic energy is thus given by

$$T \ = \ \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}I\dot{\alpha}^2 = \tfrac{1}{2}M(R-a)^2\dot{\varphi}^2 + \frac{1}{4}Ma^2\left(\frac{R\dot{\varphi}}{a}\right)^2 = \tfrac{1}{2}\mu\dot{\varphi}^2$$

$$\mu = M\left((R-a)^2 + \frac{R^2}{2}\right)$$

The potential energy is $V = MgY$, so that the Lagrangian is

$$L = \tfrac{1}{2}\mu\dot{\varphi}^2 + Mg(R-a)\cos\varphi$$

b) The canonical momentum for φ is

$$p = \frac{\partial L}{\partial \dot{\varphi}} = \mu\dot{\varphi} \implies \dot{\varphi} = p/\mu$$

Thus the Hamiltonian is

$$H = p\dot{\varphi} - L = \frac{p^2}{2\mu} - Mg(R-a)\cos\varphi$$

The canonical equations of motion are given by

$$\dot{\varphi} = \frac{\partial H}{\partial p} = p/\mu, \qquad \dot{p} = -\frac{\partial H}{\partial \varphi} = -Mg(R-a)\sin\varphi$$

These may be combined as

$$\ddot{\varphi} = -\frac{Mg(R-a)}{\mu}\sin\varphi$$

c) For small amplitudes oscillations, this reduces to

$$\ddot{\varphi} \approx -\omega^2\,\varphi, \qquad \omega^2 = \frac{Mg(R-a)}{\mu}$$

For $a = R/3$, $\mu = (17/18)MR^2$, $R - a = (2/3)R$, so that

$$\omega = \sqrt{\frac{12\,g}{17\,R}}$$

Problem 2 (*10 points*)

A particle of mass m moves in three dimensions in a central potential as described by the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} - \frac{V_0}{(a+r)^2}$$

where V_0, a are positive constants.

a) Write down the Hamilton-Jacobi equation and solve it by separation of variables, expressing your answer in terms of an integral over r. (This integral is not easy to do, so you do not have to do it at this stage.)

b) Now consider the special case of purely radial motion, setting angular motion to zero. Solve for the trajectory of the particle both for $E > 0$ and $E < 0$. (*Hint*: The substitution $u = \tfrac{1}{2}(a+r)^2$ will reduce the integrals you encounter to integrals of elementary functions.)

Solution

a) The Hamilton-Jacobi equation becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \frac{V_0}{(a+r)^2} = 0$$

The coefficients are independent of t and φ, so we can do a separation of variables by

$$S = W(r) - Et + l\varphi$$

The equation then reduces to

$$(W')^2 = 2mE - \frac{l^2}{r^2} + \frac{2mV_0}{(a+r)^2} \implies W' = \pm\sqrt{2mE - \frac{l^2}{r^2} + \frac{2mV_0}{(a+r)^2}}$$

The full solution is thus

$$S = -Et + l\varphi \pm \int dr \sqrt{2mE - \frac{l^2}{r^2} + \frac{2mV_0}{(a+r)^2}}$$

b) For the special case of radial motion, we can set $l = 0$, to get

$$S = -Et \pm \int dr \sqrt{2mE + \frac{2mV_0}{(a+r)^2}}$$

Differentiating with respect to the constant of integration E and setting it to a constant $-t_0$, we get

$$t - t_0 = \pm \int dr \frac{m}{\sqrt{2mE + \frac{2mV_0}{(a+r)^2}}} = \pm m \int dr \frac{(a+r)}{\sqrt{2mE(a+r)^2 + 2mV_0}}$$

Introducing $u = \frac{1}{2}(a+r)^2$, we can write this as

$$t - t_0 = m \pm \int \frac{du}{\sqrt{4mEu + 2mV_0}} = \pm\frac{1}{2E}\sqrt{4mEu + 2mV_0}$$

Squaring this we get

$$\frac{V_0}{2E} + u = \frac{E(t_0 - t)^2}{m}$$

For $E < 0$, write $E = -\epsilon$, $\epsilon > 0$. Then we get

$$a + r = \pm\sqrt{\frac{V_0}{\epsilon} - \frac{2\epsilon(t_0 - t)^2}{m}}$$

For $E > 0$, the solution is

$$a + r = \pm\sqrt{\frac{2E(t_0 - t)^2}{m} - \frac{V_0}{E}}$$

Depending on initial conditions, we may need to choose t_0 as either positive or negative. In nay case, it can be checked that the particle always falls in to the center.

Problem 3 (*12 points*)

Consider the dynamics of a particle of mass m in three dimensions. We define the quantities

$$D_1 = \tfrac{1}{2}p^2, \quad D_2 = \tfrac{1}{2}x^2, \quad D_3 = \vec{x} \cdot \vec{p}$$

a) Calculate the Poisson brackets $\{D_i, D_j\}$ and show that they form a closed algebra.
b) Now consider the Hamitlonian

$$H = \frac{p^2}{2m} - \frac{\kappa}{r^2}$$

Consider the time-evolution of D_3 as defined by Poisson bracket with H and solve this equation for the case of a trajectory of fixed energy E.

Solution
From the definition of the Poisson brackets,

$$\{D_1, D_2\} = \sum_i \left(\frac{\partial D_1}{\partial x^i} \frac{\partial D_2}{\partial p_i} - \frac{\partial D_1}{\partial p_i} \frac{\partial D_2}{\partial x^i} \right) = -x_i p_i = -D_3$$

$$\{D_3, D_1\} = \sum_i p_i p_i = 2D_1, \qquad \{D_3, D_2\} = -x_i x_i = -2D_2$$

Since the result of the Poisson bracket is expressed back in terms of the D's, we see that the algebra of the three functions D_i is closed under the Poisson bracket operation.
b) The time evolution of any function is given by its Poisson bracket with H. We will need the Poisson bracket

$$\{D_3, \frac{1}{r^2}\} = -x_i \frac{\partial}{\partial x^i} \frac{1}{r^2} = x_i \frac{2x_i}{r^4} = \frac{2}{r^2}$$

Thus we get

$$\frac{\partial D_3}{\partial t} = \{D_3, H\} = \frac{1}{m}\{D_3, D_1\} - \kappa\{D_3, \frac{1}{r^2}\} = \frac{2D_1}{m} - \kappa\frac{2}{r^2} = 2H$$

For trajectories with energy E, we have $H = E$ and we can integrate this to get

$$D_3(t) = D_3(0) - 2Et$$

Problem 4 (*10 points*)

a) Consider the radiation of a photon (viewed as a particle of zero mass) by a charged particle of mass m (such as an electron) moving with a constant velocity. Show by conservation of energy and momentum, that this process is forbidden in the vacuum.
b) The Lorentz transformation of the momentum of a particle of mass m is given, for small values of the velocity change for the frame of reference (denoted by ϵ_i), as

$$p_0' \approx p_0 + \epsilon_i p_i, \qquad p_i' \approx p_i + \epsilon_i p_0, \qquad p_0 = \sqrt{p^2 + m^2 c^2}$$

Show that the changes in the momentum components can be given by their Poisson bracket with $-\epsilon_i K_i$ with $K_i = x_i \sqrt{p^2 + m^2 c^2}$.

Solution

a) Let k^μ denote the 4-momentum of the photon and let p^μ and p'^μ denote the 4-momenta of the particle before and after the emission of the photon. For these momenta, we have the results

$$\eta_{\mu\nu} p^\mu p^\nu = \eta_{\mu\nu} p'^\mu p'^\nu = m^2 c^2, \qquad \eta_{\mu\nu} k^\mu k^\nu = 0$$

where m is the mass of the charged particle and the second equality indicates that the photon is massless. By conservation of energy and momentum, we have

$$p^\mu = p'^\mu + k^\mu$$

Taking k^μ to the left hand side and squaring with the metric $\eta_{\mu\nu}$, we find

$$\eta_{\mu\nu} p^\mu p^\nu + \eta_{\mu\nu} k^\mu k^\nu - 2\eta_{\mu\nu} p^\mu k^\nu = \eta_{\mu\nu} p'^\mu p'^\nu$$

This reduces to

$$\eta_{\mu\nu} p^\mu k^\nu = 0 \implies p_0 k_0 = \vec{p} \cdot \vec{k} = |\vec{p}| k_0 \cos\theta$$

Since $p_0 = \sqrt{\vec{p}^2 + m^2 c^2}$, this condition will require that

$$\cos\theta = \frac{p_0}{|\vec{p}|} = \frac{\sqrt{\vec{p}^2 + m^2 c^2}}{|\vec{p}|} > 1$$

Since $\cos\theta \leq 1$ for real angles, this requirement is impossible showing that the emission of the photon by the electron (without additional accelerating forces) is impossible.

b) The change of any function f generated by a function G is given as $\delta f = \{f, G\}$. Taking $G = -\epsilon_i K_i = -\epsilon_i x_i \sqrt{p^2 + m^2 c^2}$, we get

$$\{p_i, G\} = -\sum_k \left(\frac{\partial p_i}{\partial x^k} \frac{\partial}{\partial p_k} \epsilon \cdot K - \frac{\partial p_i}{\partial p_k} \frac{\partial}{\partial x^k} \epsilon \cdot K \right) = \delta_{ik} \epsilon_k \sqrt{p^2 + m^2 c^2} = \epsilon_i \, p_0$$

$$\{p_0, G\} = -\sum_k \left(\frac{\partial p_0}{\partial x^k} \frac{\partial}{\partial p_k} \epsilon \cdot K - \frac{\partial p_0}{\partial p_k} \frac{\partial}{\partial x^k} \epsilon \cdot K \right) = \frac{p_k}{p_0} \epsilon_k p_0 = \epsilon_i p_i$$

This shows that

$$p'_i = p_i + \{p_i, G\}, \qquad p'_0 = p_0 + \{p_0, G\}$$

Problem 5 (Bonus, *8 points*)

The Lagrangian describing the motion of a spinning top placed on the ground is given in terms of the three Euler angles as

$$L = \tfrac{1}{2} I_1 \left(\dot\theta^2 + \dot\psi^2 \sin^2\theta \right) + \tfrac{1}{2} I_3 \left(\dot\varphi + \dot\psi \cos\theta \right)^2 - Mgh \cos\theta$$

Obtain the canonical momenta and the Hamiltonian (in terms of the coordinates and canonical momenta).

<u>Solution</u>
The canonical momenta are given by

$$
\begin{aligned}
p_\theta &= \frac{\partial L}{\partial \dot\theta} = I_1 \dot\theta \\
p_\psi &= \frac{\partial L}{\partial \dot\psi} = I_1 \sin^2\theta\, \dot\psi + I_3 \cos\theta(\dot\varphi + \dot\psi\cos\theta) \\
p_\varphi &= \frac{\partial L}{\partial \dot\varphi} = I_3(\dot\varphi + \dot\psi\cos\theta)
\end{aligned}
$$

Using the last of these, the first two can be solved for the velocities as

$$
\dot\theta = \frac{p_\theta}{I_1}, \qquad \dot\psi = \frac{p_\psi - \cos\theta\, p_\varphi}{I_1 \sin^2\theta}, \qquad \dot\varphi = \frac{p_\varphi}{I_3} - \cos\theta\,\dot\psi = \frac{p_\varphi}{I_3} - \cos\theta\left[\frac{p_\psi - \cos\theta\, p_\varphi}{I_1 \sin^2\theta}\right]
$$

The Hamiltonian is obtained as

$$
\begin{aligned}
H &= p_\theta\dot\theta + p_\psi\dot\psi + p_\varphi\dot\varphi - L \\
&= p_\theta\frac{p_\theta}{I_1} - \tfrac{1}{2}I_1\left(\frac{p_\theta}{I_1}\right)^2 + p_\varphi\left(\frac{p_\varphi}{I_3} - \dot\psi\cos\theta\right) - \tfrac{1}{2}I_3\left(\frac{p_\varphi}{I_3}\right)^2 + p_\psi\dot\psi - \tfrac{1}{2}I_1\sin^2\theta\,\dot\psi^2 + Mgh\cos\theta \\
&= \frac{p_\theta^2}{2I_1} + \frac{p_\varphi^2}{2I_3} + (p_\psi - p_\varphi\cos\theta)\dot\psi - \tfrac{1}{2}I_1\sin^2\theta\,\dot\psi^2 + Mgh\cos\theta \\
&= \frac{p_\theta^2}{2I_1} + \frac{p_\varphi^2}{2I_3} + \frac{(p_\psi - p_\varphi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta
\end{aligned}
$$

Useful results
General

$$
\begin{aligned}
ds^2 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 & \text{(Cartesian coordinates)} \\
&= dr^2 + r^2 d\varphi^2 + dz^2 & \text{(Cylindrical coordinates)} \\
&= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2 & \text{(Spherical polar coordinates)} \\
L &= T - V, \qquad T = \tfrac{1}{2}m(\dot x_1^2 + \dot x_2^2 + \dot x_3^2), \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}
\end{aligned}
$$

Rotating frame, Euler equations

$$
L = \frac{m}{2}\left(\dot x^2 - 2\,\epsilon_{ijk}\dot x^i x^j \omega^k + \omega^2 x^2 - (\vec\omega \cdot \vec x)^2\right) - V(x)
$$

$$
\begin{aligned}
I_1\dot\Omega_1 + (I_3 - I_2)\Omega_2\Omega_3 &= \tau_1 \\
I_2\dot\Omega_2 + (I_1 - I_3)\Omega_3\Omega_1 &= \tau_2
\end{aligned}
$$

$$I_3 \dot{\Omega}_3 + (I_2 - I_1)\Omega_1 \Omega_2 \;=\; \tau_2$$

Moment of inertia

$$I = \frac{2}{5}MR^2 \;\; \text{(Sphere)}, \qquad I = \frac{1}{2}MR^2 \;\; \text{(Disk around its axis)}$$

Canonical formulation

$$p_i \;=\; \frac{\partial L}{\partial \dot{q}_i}, \qquad H(p,q) = p_i \dot{q}_i - L$$

$$\dot{q}_i \;=\; \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \frac{\partial f}{\partial t} = \{f, H\}$$

$$\{f, h\} \;=\; \sum_k \left(\frac{\partial f}{\partial q^k} \frac{\partial h}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial h}{\partial q^k} \right)$$

Relativity

4-momentum $p_\mu = (p_0, p_i)$, $p_0 = H/c$.

$$\eta_{\mu\nu} p^\mu p^\nu = p_0^2 - \vec{p} \cdot \vec{p} = m^2 c^2, \qquad p_0 = \sqrt{\vec{p}^2 + m^2 c^2}$$